

THE WAHL MAP OF ONE-NODAL CURVES ON K3 SURFACES

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ABSTRACT. We consider a general primitively polarized K3 surface (S, H) of genus $g + 1$ and a 1-nodal curve $\tilde{C} \in |H|$. We prove that the normalization C of \tilde{C} has surjective Wahl map provided $g = 40, 42$ or ≥ 44 .

To L. Ein on the occasion of his 60-th birthday

INTRODUCTION

In this Note we consider 1-nodal curves lying on a K3 surface and we study the gaussian map, or Wahl map, on their normalization. If we consider a primitive linear system $|H|$ on a K3 surface S , then it is well known that every nonsingular $\tilde{C} \in |H|$ has a non-surjective Wahl map

$$\Phi_K : \bigwedge^2 H^0(C, \omega_C) \longrightarrow H^0(C, \omega_C^3)$$

(see §1 for the definition) and that, if moreover $\tilde{C} \in |H|$ is general then it is Brill-Noether-Petri general [W2, L]. It is of some interest to decide whether the same properties hold for the normalization C of a 1-nodal $\tilde{C} \in |H|$, and more generally for the normalization of a singular \tilde{C} in $|H|$. The Brill-Noether theory of singular curves on a K3 surface has received quite a lot of attention in recent times, see e.g. [Go, FKP, BFT, CK, Ke]. On the other hand to our knowledge very little is known on their Wahl map. In [Ke] and [H2] the authors consider a modified version of the Wahl map, which does not seem to have a direct and simple relation with the ordinary Wahl map Φ_K ; in particular their results point towards the non-surjectivity of such modified map. On the other hand in [FKPS] it is proved that the normalization C (of genus 10) of a general 1-nodal curve $\tilde{C} \in |H|$ on a general polarized (S, H) of genus 11 has general moduli; then the

main result of [CHM] implies that Φ_K is surjective for such a curve. This surjectivity result is extended in the present paper in the following form:

Theorem 1. *Let (S, H) be a general primitively polarized K3 surface of genus $g + 1$. Assume that $g = 40, 42$ or ≥ 44 . Let $\tilde{C} \in |H|$ be a 1-nodal curve and C its normalization. Then the Wahl map*

$$\Phi_K : \bigwedge^2 H^0(C, \omega_C) \longrightarrow H^0(C, \omega_C^3)$$

is surjective.

This of course gives yet another proof of the main result of [CHM] for the values of g as in the statement because 1-nodal curves are known to exist in $|H|$ for a general primitively polarized (S, H) of any genus $g + 1 \geq 2$ [MM, Ch].

Now a few words about the method of proof. Letting $P \in S$ be the unique singular point of \tilde{C} we consider the blow-up $\sigma : X := \text{Bl}_P S \longrightarrow S$ at P and we let $E \subset X$ be the exceptional curve. Then the normalization of \tilde{C} is the strict transform $C = \sigma^* \tilde{C} - 2E \subset X$. The Wahl map Φ_K on C can be decomposed as

$$\Phi_K = H^0(\rho) \cdot \Phi_{K_X+C}$$

where Φ_{K_X+C} is a gaussian map on X and $H^0(\rho)$ is induced in cohomology by a restriction homomorphism:

$$\rho : \Omega_X^1(2K_X + 2C) \longrightarrow \omega_C^3$$

on X . We study these two maps and prove their surjectivity separately. This method of proof is analogous to the one adopted in the work of several authors before, notably [W1, DM, CLM1, CLM2]. The restriction on the genus depends on the proof: one would expect the result to hold for $g = 10$ (as it does indeed, as already remarked) and for $g \geq 12$. In fact the surjectivity of $H^0(\rho)$ holds for $g = 10$ or $g \geq 12$ (see Lemma 2). On the other hand the proof of the surjectivity of Φ_{K_X+C} , which consists in adapting an analogous proof given in [CLM2] for plane curves, leads to the restrictions on g : in fact this proof requires that we decompose a certain divisor on X as the sum of three very ample ones and this decomposition forces the genus to increase.

Recent work by M. Kemeny [Ke] implies that the curves C considered here, i.e. normalizations of 1-nodal curves on a general primitive K3 surface, are generically Brill-Noether-Petri general and fill a locus in \mathcal{M}_g , the coarse moduli space of curves of genus g , whose closure has

dimension $19+g$. Theorem 1 and [W2] imply that this naturally defined locus is not contained in the so-called K3-locus (i.e. the locus of smooth curves that can be embedded in a K3 surface).

One can ask whether a result analogous to Theorem 1 can be proved for the normalization of curves on K3 surfaces having a more complicated singular point. We did not consider this case. Note though that, to our knowledge, such curves are known to exist only in the case of A_k -singularities or ordinary triple points (see [GK, Ga]).

The paper is organized as follows. In §1 we introduce the gaussian maps and explain the strategy of proof of the surjectivity of the Wahl map of a curve lying in a regular surface. In §2 we prove the surjectivity of $H^0(\rho)$ and in §3 we prove the surjectivity of Φ_{K_X+C} . We work over \mathbb{C} .

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1. GENERALITIES ON GAUSSIAN MAPS

In this section we recall a few definitions and basic facts concerning gaussian maps. Given line bundles L, M on a nonsingular projective variety Y we consider:

$$\mathcal{R}(L, M) = \ker[H^0(Y, L) \otimes H^0(Y, M) \rightarrow H^0(Y, L \otimes M)]$$

Then we have a canonical map:

$$\Phi_{L,M} : \mathcal{R}(L, M) \longrightarrow H^0(Y, \Omega_X^1 \otimes L \otimes M)$$

called the *gaussian map*, or *Wahl map*, of L, M , which is defined as follows. Let $\Delta \subset Y \times Y$ be the diagonal and $p_1, p_2 : Y \times Y \rightarrow Y$ the projections. Then

$$\mathcal{R}(L, M) = H^0(Y \times Y, p_1^*L \otimes p_2^*M \otimes \mathcal{I}_\Delta)$$

Since $\mathcal{I}_\Delta \otimes \mathcal{O}_\Delta = \Omega_Y^1$, the restriction to Δ :

$$p_1^*L \otimes p_2^*M \otimes \mathcal{I}_\Delta \longrightarrow p_1^*L \otimes p_2^*M \otimes I_\Delta \otimes \mathcal{O}_\Delta$$

induces $\Phi_{L,M}$ on global sections. The exact sequence:

$$0 \longrightarrow p_1^*L \otimes p_2^*M \otimes \mathcal{I}_\Delta^2 \longrightarrow p_1^*L \otimes p_2^*M \otimes \mathcal{I}_\Delta \longrightarrow p_1^*L \otimes p_2^*M \otimes I_\Delta \otimes \mathcal{O}_\Delta \longrightarrow 0$$

shows that the vanishing:

$$H^1(Y \times Y, p_1^*L \otimes p_2^*M \otimes \mathcal{I}_\Delta^2) = 0$$

is a sufficient condition for the surjectivity of $\Phi_{L,M}$.

In case $L = M$ we have $\mathcal{R}(L, L) = I_2(Y) \oplus \bigwedge^2 H^0(Y, L)$, where

$$I_2(Y) = \ker[S^2 H^0(Y, L) \rightarrow H^0(Y, L^2)]$$

and $\Phi_{L,L}$ is zero on $I_2(Y)$. Therefore $\Phi_{L,L}$ is equivalent to its restriction to $\bigwedge^2 H^0(Y, L)$, which is denoted by

$$\Phi_L : \bigwedge^2 H^0(Y, L) \longrightarrow H^0(Y, \Omega_X^1 \otimes L^2)$$

In particular, for a non-hyperelliptic curve C we are interested in $\Phi_{K,K}$ or rather in

$$\Phi_K : \bigwedge^2 H^0(C, \omega_C) \longrightarrow H^0(C, \omega_C^3)$$

where $\mathcal{O}_C(K) = \omega_C$ is the canonical invertible sheaf.

Suppose that $C \subset X$ where X is a nonsingular regular surface. Then $H^0(C, \omega_C) = H^0(X, K_X + C)$ and Φ_K can be described as a composition:

$$(1) \quad \Phi_K = H^0(\rho) \cdot \Phi_{K_X+C}$$

where

$$\Phi_{K_X+C} : \bigwedge^2 H^0(C, \omega_C) = \bigwedge^2 H^0(X, \mathcal{O}_X(K_X+C)) \longrightarrow H^0(X, \Omega_X^1(2K_X+2C))$$

and $\rho : \Omega_X^1(2K_X+2C) \rightarrow \omega_C^3$ is the restriction map. Therefore if both $H^0(\rho)$ and Φ_{K_X+C} are surjective, the surjectivity of Φ_K will follow. This is how we will prove it in our situation.

2. THE SURJECTIVITY OF $H^0(\rho)$

As in the Introduction, we let (S, H) be a K3 surface with a polarization of genus $g+1 \geq 3$ and let $\tilde{C} \in |H|$ be a curve with one node, i.e. an ordinary double point, at $P \in S$ and no other singularities. Consider the blow-up $\sigma : X := \text{Bl}_P S \rightarrow S$ of S at P , let $E \subset X$ be the exceptional curve and $C = \sigma^* \tilde{C} - 2E \subset X$ the strict transform of \tilde{C} . Then we have a homomorphism of sheaves

$$\rho : \Omega_X^1(2K_X + 2C) \longrightarrow \omega_C^3$$

induced by the natural restriction $\Omega_X^1 \rightarrow \omega_C$. We prove:

Lemma 2. *Suppose that (S, H) is a general primitively polarized K3 surface of genus $g+1$, with $g = 10$ or $g \geq 12$. Then, with the notations introduced above, the map:*

$$H^0(\rho) : H^0(X, \Omega_X^1(2K_X + 2C)) \longrightarrow H^0(C, \omega_C^3)$$

is surjective.

Proof. We have an exact sequence on X :

$$0 \longrightarrow \Omega_X^1(\log C)(-C) \longrightarrow \Omega_X^1 \longrightarrow \omega_C \longrightarrow 0$$

where $\Omega_X^1(\log C)$ is the sheaf of 1-forms with logarithmic poles along C [EV]. Tensoring with $\mathcal{O}(2K_X + 2C)$ we obtain:

$$0 \longrightarrow \Omega_X^1(\log C)(2K_X + C) \longrightarrow \Omega_X^1(2K_X + 2C) \xrightarrow{\rho} \omega_C^3 \longrightarrow 0$$

Since $K_X = E$ we have $\mathcal{O}_X(2K_X + C) = \sigma^*H$ and therefore in order to prove the Lemma it suffices to prove that

$$(2) \quad H^1(X, \Omega_X^1(\log C) \otimes \sigma^*H) = 0$$

Consider the relative cotangent sequence of σ :

$$0 \longrightarrow \sigma^*\Omega_S^1 \longrightarrow \Omega_X^1 \xrightarrow{\eta} \omega_E \longrightarrow 0$$

and tensor it by σ^*H :

$$0 \longrightarrow \sigma^*[\Omega_S^1 \otimes H] \longrightarrow \Omega_X^1 \otimes \sigma^*H \xrightarrow{\eta_H} \omega_E \longrightarrow 0$$

We have:

$$H^2(X, \sigma^*[\Omega_S^1 \otimes H]) = H^2(S, \Omega_S^1 \otimes H) = H^0(S, T_S \otimes H^{-1})^\vee = 0$$

From the assumption about the genus and from the generality of (S, H) it follows that we also have:

$$H^1(X, \sigma^*[\Omega_S^1 \otimes H]) = H^1(S, \Omega_S^1 \otimes H) = 0$$

(see [B], 5.2). Therefore η_H induces an isomorphism:

$$H^1(\eta_H) : H^1(X, \Omega_X^1 \otimes \sigma^*H) \cong H^1(E, \omega_E) \cong \mathbb{C}$$

Then from the exact sequence:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega_X^1 \otimes \sigma^*H & \longrightarrow & \Omega_X^1(\log C) \otimes \sigma^*H & \longrightarrow & \sigma^*H \otimes \mathcal{O}_C \longrightarrow 0 \\ & & & & & & \parallel \\ & & & & & & \omega_C(E) \end{array}$$

we see that, since $H^1(C, \omega_C(E)) = 0$, in order to prove (2) it suffices to show that the coboundary map:

$$(3) \quad \partial_H : H^0(C, \omega_C(E)) \longrightarrow H^1(X, \Omega_X^1 \otimes \sigma^*H)$$

is non-zero.

We can argue as follows. Consider the commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \Omega_X^1 & \longrightarrow & \Omega_X^1(\log C) & \longrightarrow & \mathcal{O}_C \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Omega_X^1 \otimes \sigma^* H & \longrightarrow & \Omega_X^1(\log C) \otimes \sigma^* H & \longrightarrow & \omega_C(E) \longrightarrow 0
\end{array}$$

where the vertical maps are induced by multiplication with a section $0 \neq s \in H^0(X, \sigma^* H)$. Then we have the following commutative diagram involving the coboundary maps of both sequences:

$$\begin{array}{ccccc}
H^0(C, \mathcal{O}_C) & \xrightarrow{\partial} & H^1(X, \Omega_X^1) & & \\
\downarrow s & & \downarrow s & \searrow H^1(\eta) & \\
H^0(C, \omega_C(E)) & \xrightarrow{\partial_H} & H^1(X, \Omega_X^1 \otimes \sigma^* H) & \nearrow H^1(\eta_H) & H^1(E, \omega_E)
\end{array}$$

and it suffices to prove that

$$H^1(\eta) \cdot \partial : H^0(C, \mathcal{O}_C) \rightarrow H^1(E, \omega_E)$$

is non-zero.

Observe that ∂ associates to $1 \in H^0(C, \mathcal{O}_C)$ the Atiyah-Chern class of $\mathcal{O}_X(C)$ and $H^1(\eta)(\partial(1))$ is its restriction to E . We have that $\ker(H^1(\eta)) \cong H^1(X, \sigma^* \Omega_S^1)$ is generated by the Atiyah-Chern classes of curves coming from S , which are trivial when restricted to E . Since $E \cdot C = 2$ we see that $\partial(1) \notin \ker(H^1(\eta))$ and this proves the Lemma. \square

3. THE GAUSSIAN MAP ON X

We keep the notations of §2. We will prove the following:

Proposition 3. *Suppose that (S, H) is a general primitively polarized K3 surface of genus $g + 1$, with $g = 40, 42$ or ≥ 44 ; let $\tilde{C} \in |H|$ be a 1-nodal curve and C is normalization. Let $X = \text{Bl}_P(S)$ where $P \in \tilde{C}$ is the node. Then the gaussian map*

$$\Phi_{K_X+C} : \bigwedge^2 H^0(X, \mathcal{O}_X(K_X + C)) \longrightarrow H^0(X, \Omega_X^1(2K_X + 2C))$$

is surjective.

We will need the following result and its corollary:

Proposition 4. *a) For every $d \geq 10$ there is a K3 surface S containing two very ample nonsingular curves A, B such that $\text{Pic}(S) = \mathbb{Z}[A] \oplus \mathbb{Z}[B]$, with intersection matrix:*

$$\begin{pmatrix} A^2 & A \cdot B \\ B \cdot A & B^2 \end{pmatrix} = \begin{pmatrix} 8 & d \\ d & 8 \end{pmatrix}$$

and A and B non-trigonal.

b) For every $d \geq 12$ there is a K3 surface S containing two very ample nonsingular curves A, B such that $\text{Pic}(S) = \mathbb{Z}[A] \oplus \mathbb{Z}[B]$, with intersection matrix:

$$\begin{pmatrix} A^2 & A \cdot B \\ B \cdot A & B^2 \end{pmatrix} = \begin{pmatrix} 10 & d \\ d & 8 \end{pmatrix}$$

and A and B non-trigonal.

In both cases (a) and (b) the surface S does not contain rational nonsingular curves R such that $A \cdot R = 1$ or $B \cdot R = 1$.

Proof. The Proposition is a special case of [Kn], Theorem 4.6. We obtain the Proposition by taking (with the notations used there) $(n, g-1) = (4, 4)$ and $(n, g-1) = (5, 4)$ respectively. The restriction on d is forced by the requirement that the hypotheses of the theorem apply symmetrically w.r. to A and B so that both are very ample. The non-trigonality follows from the fact that S is embedded by both $|A|$ and $|B|$ so to be an intersection of quadrics. The last assertion is proved as done in loc. cit. for (-2) curves, by comparing discriminants. \square

Corollary 5. *If S is as in Prop. 4(a) then $H = A + 2B$ defines a primitive very ample divisor class of genus $g+1 = 21 + 2d$ for every $d \geq 10$. If S is as in Prop. 4(b) then $H = A + 2B$ defines a primitive very ample divisor class of genus $g+1 = 22 + 2d$ for every $d \geq 12$.*

Proof. Clearly H is very ample and it is primitive because the generator A of $\text{Pic}(S)$ appears with coefficient 1. The genus in either case is readily computed using the intersection matrix. \square

Proof. of Proposition 3. By semicontinuity it suffices to prove the Proposition for just one primitively polarized K3 surface for each value of g as in the statement. We take (S, H) as in Corollary 5, distinguishing cases (a) and (b) according to the parity of g . Letting \tilde{C}, C, P and X as in the statement, consider the product $X \times X$ and the blow-up $\pi : Y = \text{Bl}_\Delta(X \times X) \rightarrow X \times X$ along the diagonal Δ . Let $\Lambda \subset Y$ be the exceptional divisor. For any coherent sheaf \mathcal{F} on X we define $\mathcal{F}_i = (p_i \cdot \pi)^* \mathcal{F}$, $i = 1, 2$. It suffices to prove that

$$H^1(X \times X, p_1^* \mathcal{O}_X(K_X + C) \otimes p_2^* \mathcal{O}_X(K_X + C) \otimes \mathcal{I}_\Delta^2) = 0$$

which is equivalent to:

$$H^1(Y, (K_X + C)_1 + (K_X + C)_2 - 2\Lambda) = 0$$

Note that we have:

$$(K_X + C)_1 + (K_X + C)_2 - 2\Lambda = K_Y + C_1 + C_2 - 3\Lambda$$

and therefore we want to prove that:

$$(4) \quad H^1(Y, \mathcal{L}) = 0$$

where we set $\mathcal{L} = \mathcal{O}_Y(K_Y + C_1 + C_2 - 3\Lambda)$. The proof is an adaptation of the proofs of Lemmas (3.1) and (3.10) of [CLM2]. One uses the following:

Lemma 6. *Assume that D is a very ample divisor on X . Then $D_1 + D_2 - \Lambda$ is big and nef on Y .*

Proof. See [BEL], Claim 3.3. □

Let $M := \sigma^*H(-3E)$ on X . We have:

$$M = \sigma^*A(-E) + 2(\sigma^*B(-E))$$

Since A and B are non-trigonal S has no trisecant lines through P and does not contain a line through P whether it is embedded by $|A|$ or by $|B|$ (Proposition 4). Therefore every curvilinear subscheme of S of length 3 containing P imposes independent conditions to both $|A|$ and $|B|$. Then it follows from [Co], Prop. 1.3.4, that both $\sigma^*A(-E)$ and $\sigma^*B(-E)$ are very ample on X . Therefore by the Lemma we have that both $\sigma^*A(-E)_1 + \sigma^*A(-E)_2 - \Lambda$ and $\sigma^*B(-E)_1 + \sigma^*B(-E)_2 - \Lambda$ are big and nef. It follows that the divisor $M_1 + M_2 - 3\Lambda$ is big and nef, being the sum of three big and nef divisors.

Since $C \sim M + E$ we have $C_1 + C_2 \sim M_1 + M_2 + E_1 + E_2$ and therefore we have an exact sequence on Y :

$$0 \rightarrow \mathcal{O}_Y(K_Y + M_1 + M_2 - 3\Lambda) \rightarrow \mathcal{L} \rightarrow \mathcal{O}_{E_1+E_2}(\mathcal{L}) \rightarrow 0$$

By Kawamata-Vieweg we have $H^1(Y, K_Y + M_1 + M_2 - 3\Lambda) = 0$: therefore in order to prove (4) it suffices to show that

$$(5) \quad H^1(E_1 + E_2, \mathcal{O}_{E_1+E_2}(\mathcal{L})) = 0$$

Letting $W := E_1 \cap E_2$ we have an exact sequence:

$$0 \rightarrow \mathcal{O}_{E_1}(\mathcal{L} - W) \rightarrow \mathcal{O}_{E_1+E_2}(\mathcal{L}) \rightarrow \mathcal{O}_{E_2}(\mathcal{L}) \rightarrow 0$$

and, by symmetry, it suffices to prove that:

$$(6) \quad H^1(E_1, \mathcal{O}_{E_1}(\mathcal{L} - W)) = 0$$

and

$$H^1(E_1, \mathcal{O}_{E_1}(\mathcal{L})) = 0$$

We can then consider the exact sequence on E_1 :

$$0 \rightarrow \mathcal{O}_{E_1}(\mathcal{L} - W) \rightarrow \mathcal{O}_{E_1}(\mathcal{L}) \rightarrow \mathcal{O}_W(\mathcal{L}) \rightarrow 0$$

and finally we are reduced to prove (6) and

$$(7) \quad H^1(W, \mathcal{O}_W(\mathcal{L})) = 0$$

Let $U \cong E \times E$ be the proper transform of $E \times E$ in Y . Then in E_1 we have $W = U + \Lambda|_{E_1}$. As in [CLM2], proof of Lemma (3.1), one shows that $\Lambda|_{E_1} \cong \mathbb{P}\mathcal{E}$, where $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$, and $\mathcal{L}|_{\mathbb{P}\mathcal{E}} = \mathcal{O}_{\mathbb{P}\mathcal{E}}(2C_0 + 2f)$ where $C_0 \in |\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)|$ and f is a fibre of $\mathbb{P}\mathcal{E} \rightarrow \mathbb{P}^1$. Now the proof proceeds as in [CLM2], Lemma (3.1), after having proved that $\mathcal{O}_{E_1}(\mathcal{L} - W) = \mathcal{O}_{E_1}(K_Y + M_1 + M_2 - 3\Lambda)$ is big and nef. This last fact is obtained exactly as in the proof of Lemma (3.10) of [CLM2], using the fact that both $\sigma^*A(-E)$ and $\sigma^*B(-E)$ are very ample on X . \square

Proof. of Theorem 1. Recalling (1), the theorem follows immediately from Lemma 2 and from Proposition 3. \square

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